Multipopulation Replicator Dynamics with Erroneous Perceptions*

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Abstract – *In evolutionary game theory, to the best of our knowledge, individuals' perceptions have been disregarded. When an individual interacts with the other individual under coexistence of heterogeneous sub-populations, it is natural that the individual changes its strategy depending on the sub-population the other individual belongs to. Moreover, in such a situation, each individual may make an error about the sub-population the other individual belongs to. In this paper, we propose a multipopulation model with erroneous perceptions. We define an evolutionarily stable strategy (ESS) and formulate replicator dynamics in this model, and prove several properties of the proposed model.*

Keywords: Evolutionary game, replicator dynamics, perception.

1 Introduction

In evolutionary game theory, the distribution of strategies in the population is changed according to payoffs which individuals earn depending on their selected strategies [7]. Its extensions to asymmetric games have been studied in [2] [3] [5].

It is assumed in asymmetric games that each individual interacts once per unit time with one individual drawn from each sub-population randomly. This assumption indicates that individuals can identify which sub-populations the other individuals belong to. When an individual interacts with the other individual, it is natural that the individual changes its strategy depending on the sub-population the other individual belongs to.

Moreover, in complex conflict situations, it is natural that individuals' perceptions are different each other and their behaviors depend on their perceptions to the game. Such a situation is modeled by a hypergame [1]. Various solution concepts and methods of analysis taking players' perceptions into account have been proposed for hypergames [6]. Recently, replicator dynamics for multipopulation hypergames has been proposed in [4]. However, in multipopulation games, each individual may make an error about the sub-population the other individual belongs to. In other words, each individual selects a strategy according to erroneous identification for the sub-population of the other individual.

Thus, in this paper, we consider that individuals change their strategies depending on the other individuals whom they play the game with. Moreover, we introduce individuals' perceptions into our model in a different way from hypergames. In our model, we consider individuals know that a population is partitioned into multiple sub-populations, but can not always correctly identify which sub-populations the other individuals belong to. Then, when individuals play the game, they judge which sub-population the other individual belongs to, and each of them selects a strategy depending on their own erroneous perceptions. We define an evolutionarily stable strategy (ESS), formulate replicator dynamics for this model, and prove several properties of the proposed model.

2 Preliminaries

2.1 Two-player Normal Form Games

Normal form games are defined by players, their strategies, and payoffs depending on selected strategies. For player i $(i = 1, 2)$, let Φ_i and S_i be sets of player i's pure and mixed strategies, respectively. In two-player games, we can write players' payoff functions as payoff matrices. Let A_i $(i = 1, 2)$ be player i's payoff matrix. If player 1 selects $s_1 \in S_1$ and player 2 selects $s_2 \in S_2$, then the payoff which the former earns is $s_1^T A_1 s_2$ and the latter earns is $s_2^T A_2 s_1$. Denoted by $\mathrm{int}(S_i)$ is the interior of S_i.

2.2 An Evolutionary Game Model of Asymmetric Games

In classical game theory, we suppose that players are fully rational. In the real world, however, it is natural that a player selects a strategy by trial and error. In evolutionary game theory, players are not supposed to be rational, and the distribution of strategies in the population is changed according to payoffs which individuals earn depending on their selected strategies [7]. Its extensions to asymmetric games have been studied [2] [3] [5].

We assume that a large population is partitioned into n sub-populations $P_1, P_2, \ldots, P_n$. Each individual is assumed to interact once per unit time with one random individual in each sub-population. For each sub-population P_i ($i = 1, 2, \cdots, n$), let $\Phi_i = \{1, 2, \cdots, m_i\}$ and S_i be sets of pure and mixed strategies of sub-population P_i, respectively. Let A_{ij} be a payoff matrix for an individual of P_i at the case of playing the game with an individual of P_j.

In an asymmetric game model, several concepts of an evolutionarily stable strategy (ESS) have been proposed [2] [3] [5]. In this paper, we employ the concept of a strict ESS proposed by Garay [3].

For all i ($i = 1, \cdots, n$), suppose that all individuals of P_i adopt the same strategy $s_i^* \in S_i$, and a small group of individuals adopting a mutant strategy $s_i \in S_i$ appears in P_i. Let ϵ_i be a proportion of mutants to all individuals in P_i. A mixed strategy combination $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \cdots \times S_n$ is an ESS if the following equation holds for all $s \neq s^*$ ($s \in S_1 \times \cdots \times S_n$), all i ($i = 1, \cdots, n$), and all sufficiently small $\epsilon_j > 0$ ($j = 1, \cdots, n$):

$$s_i^{*T} \left(\sum_{j=1}^{n} A_{ij} \bar{s}_j \right) > s_i^T \left(\sum_{j=1}^{n} A_{ij} \bar{s}_j \right), \qquad (1)$$

where $\bar{s}_j = (1 - \epsilon) s_j^* + \epsilon_j s_j$.

Replicator dynamics describes the evolution of distributions of strategies in populations. In the conventional replicator dynamics, individuals are supposed to adopt pure strategies. Let $s_i = (s_i^1, s_i^2, \ldots, s_i^{m_i})^T$ be a population state of sub-population P_i ($i = 1, 2, \cdots, n$), where s_i^k is the proportion of individuals with a pure strategy $k \in \Phi_i$ in P_i. Thus the population state is formally identical with a mixed strategy $s_i \in S_i$.

Each individual is assumed to interact once per unit time with one random individual in each sub-population. In each sub-population, suppose that the rate of increase of individuals with a strategy k is expressed as the difference between the payoffs which an individual with a strategy k earns and the average payoff the sub-population earns. Hence, replicator dynamics of this model is formulated as follows [2] [3] [5]:

$$\dot{s}_i^k = s_i^k (e_i^k - s_i)^T \left(\sum_{j=1}^{n} A_{ij} s_j \right) \qquad (2)$$

for $i = 1, \cdots, n$ and $k \in \Phi_i$, where e_i^k is the m_i-dimensional unit vector such that the kth element equals 1.

In this model, the following properties have been proved [3]. A population state corresponding to an ESS is an asymptotically stable equilibrium point of replicator dynamics (2). Moreover, if it is interior, then all orbits of replicator dynamics (2) starting from $\mathrm{int}(S_1 \times \cdots \times S_n)$ converge to the equilibrium point.

3 A Model of Asymmetric Games without Individuals' Perceptions

In this section, we assume that individuals do not know that a population is partitioned into multiple sub-populations. In order to describe such a situation, we suppose that each individual interacts once per unit time with one individual randomly drawn from all sub-populations depending on a population distribution of all sub-populations. Moreover, we assume that all individuals of the population have the same set of strategies, but the payoff matrix of each sub-population is different from each other.

Individuals of the population have the same set of pure and mixed strategies, $\Phi = \{1, 2, \cdots, m\}$ and S, respectively. We define a set of mixed strategy combinations (or population state combinations) by $S^n = \{(s_1, s_2, \cdots, s_n) | s_i \in S, i = 1, 2, \cdots, n\}$. Let A_i be a payoff matrix of P_i, and let α_i be the proportion of individuals of P_i to all individuals, where $\sum_{i=1}^{n} \alpha_i = 1$.

We consider the payoff which an individual of P_i with a strategy $s_i \in S$ earns when all individuals of sub-population P_j adopt $s_j^* \in S_j$ for each j ($j = 1, \cdots, n$), i.e. a population state combination is $s^* \in S^n$. Considering that an individual of P_i plays a game with an individual of P_j with probability α_j, the expected value of the payoff which an individual of P_i with s_i earns is $s_i^T A_i (\sum_{j=1}^{n} \alpha_j s_j^*)$. By transforming this, we have $s_i^T \sum_{j=1}^{n} (\alpha_j A_i) s_j^*$. If we adopt $\alpha_j A_i$ as a payoff matrix for P_i at the case of playing the game with P_j, then it is clear that this model is a special case of an n-population asymmetric game model given in Section 2.2. We define an ESS and formulate replicator dynamics for this model based on previous works [2] [3] [5]. We can extend an evolutionarily stable strategy (ESS) to this model.

Definition 1 A mixed strategy combination $s^* \in S^n$ is an ESS if the following equation holds for all $s \neq s^*$ ($s \in S^n$), all i ($i = 1, \cdots, n$), and all sufficiently small $\epsilon_j > 0$ ($j = 1, \cdots, n$):

$$s_i^T A_i \left(\sum_{j=1}^{n} \alpha_j \bar{s}_j \right) < s_i^{*T} A_i \left(\sum_{j=1}^{n} \alpha_j \bar{s}_j \right), \qquad (3)$$

where $\bar{s}_j = (1 - \epsilon_j) s_j^* + \epsilon_j s_j$.

We can transform Eq. (3) into

$$(s_i^* - s_i)^T A_i \left(\sum_{j=1}^{n} \alpha_j s_j^* \right) - (s_i^* - s_i)^T A_i \left\{ \sum_{j=1}^{n} \epsilon_j \alpha_j \left(s_j^* - s_j \right) \right\} > 0. \qquad (4)$$

Since Eq. (3) holds for all sufficiently small ϵ_j, there exist ϵ and ϵ_j such that $\epsilon = \epsilon_1 = \cdots = \epsilon_n$ and Eq. (4) holds. Thus, we have the following theorem:

Theorem 1 A mixed strategy combination $s^* \in S^n$ is an ESS if and only if the following two conditions hold:

1. Equilibrium condition: For all $s \in S$,

$$s_i^{*T} A_i \left(\sum_{j=1}^{n} \alpha_j s_j^* \right) \geq s_i^T A_i \left(\sum_{j=1}^{n} \alpha_j s_j^* \right). \quad (5)$$

2. Stability condition: For all $s \neq s^*$ such that the equality holds in the equilibrium condition,

$$(s_i^* - s_i)^T A_i \left(\sum_{j=1}^{n} \alpha_j s_j \right) > 0. \quad (6)$$

Suppose that individuals adopt pure strategies. Let $s_i = (s_i^1, s_i^2, \cdots, s_i^m)^T \in S$ be a population state of P_i ($i = 1, \cdots, n$), where s_i^j is the proportion of individuals with a pure strategy $j \in \Phi$ in P_i. Denoted by $s = (s_1, \cdots, s_n) \in S^n$ is a population state combination. Hence, replicator dynamics of this model is formulated as follows:

$$\dot{s}_i^j = s_i^j (e^j - s_i)^T A_i \left(\sum_{k=1}^{n} \alpha_k s_k \right) \quad (7)$$

for $i = 1, \cdots, n$ and $j \in \Phi$, where e^j is the m-dimensional unit vector such that its jth element equals 1.

As we explain in 2.2, if a mixed strategy combination $s^* \in S^n$ is an ESS, then its population state combination is an asymptotically stable equilibrium point of replicator dynamics (7).

4 Replicator Dynamics with Erroneous Perceptions

In this section, we assume that individuals know that a population is partitioned into multiple sub-populations, but can not always correctly identify which sub-populations the other individuals belong to. Then, when the individuals play the game, they judge which sub-population, $P_1, \cdots, P_n$, the other player belongs to, and select their strategies depending on their own subjective judgments. Similar to Section 3, we assume that each individual interacts once per unit time with one individual randomly drawn from all sub-populations depending on a population distribution of all sub-populations. Moreover, we assume that all individuals of the population have the same set of strategies, but the payoff matrix of each sub-population is different from each other.

Let A_i be a payoff matrix for P_i, and let α_i be the proportion of individuals of P_i to all individuals, where $\sum_{i=1}^{n} \alpha_i = 1$. Individuals of the population have the same set of pure and mixed strategies, $\Phi = \{1, 2, \cdots, m\}$ and S, respectively. Let β_i^{jk} be the probability that individuals of P_i perceive individuals of P_j as those of P_k ($i, j, k = 1, \cdots, n$, and $k \neq i$), where $\sum_{k=1}^{n} \beta_i^{jk} = 1$. Denoted by $S_i = \{(s_{i1}^T, s_{i2}^T, \cdots, s_{in}^T)^T | s_{ij} \in S, j = 1, \cdots, n\}$ is a P_i's extended set of the strategies in this model, where s_{ij} is the P_i's strategy adopted to the individuals whose sub-population is identified as P_j.

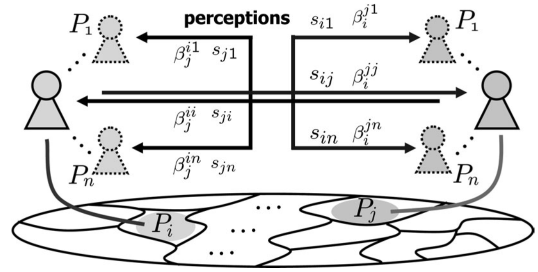

Figure 1: A model of perceptions of P_i and P_j.

We denote a set of extended strategy combinations (or population state combinations) by $S^{n \times n} = \{(s_1, \cdots, s_n) | s_i \in S_i, i = 1, \cdots, n\}$, and we define matrices E_i^j ($i, j = 1, \cdots, n$) as follows:

$$E_i^j = \left[\begin{array}{cccc} \beta_i^{j1} I_n & \beta_i^{j2} I_n & \cdots & \beta_i^{jn} I_n \end{array} \right], \quad (8)$$

where I_n is the n-dimensional unit matrix. The matrix E_i^j represents P_i's perceptions to P_j. Figure 1 shows an illustration of individuals' perceptions. Suppose that

$$s_{i1} = \cdots = s_{in}, \quad \beta_i^{jk} = \begin{cases} 0 & i \neq k, \\ 1 & i = k. \end{cases} \quad (9)$$

Then, this model is equivalent to the model of Section 3.

4.1 Evolutionarily Stable Strategy

For all i ($i = 1, \cdots, n$), suppose that all individuals of P_i adopt the same strategy $s_i^* \in S_i$, and a small group of individuals adopting a mutant strategy $s_i \in S_i$ appears in P_i. Let ϵ_i be a proportion of mutants in P_i to all individuals in P_i. Similar to Definition 1, we define an Evolutionarily Stable Strategy (ESS) as follows:

Definition 2 An extended strategy combination $s^* \in S^{n \times n}$ is an ESS if the following equation holds for all $s \neq s^*$ ($s \in S^{n \times n}$), all i ($i = 1, \cdots, n$), and all sufficiently small $\epsilon_j > 0$ ($j = 1, \cdots, n$):

$$\sum_{j=1}^{n} \alpha_j \left(E_i^j s_i \right)^T A_i (E_j^i \bar{s}_j) < \sum_{j=1}^{n} \alpha_j \left(E_i^j s_i^* \right)^T A_i (E_j^i \bar{s}_j), \quad (10)$$

where $\bar{s}_j = (1 - \epsilon_j) s_j^* + \epsilon_j s_j$.

We can rewrite Eq. (10) as

$$\sum_{j=1}^{n} \alpha_j \left\{ E_i^j (s_i^* - s_i) \right\}^T A_i \left(E_j^i s_j^* \right) - \sum_{j=1}^{n} \alpha_j \epsilon_j \left\{ E_i^j (s_i^* - s_i) \right\}^T A_i \left\{ E_j^i (s_j^* - s_j) \right\} > 0. \quad (11)$$

Thus, we have the following theorem:

Theorem 2 An extended strategy combination $s^* \in S^{n\times n}$ is an ESS if and only if the following two conditions hold:

1. Equilibrium condition: For all $s \in S^{n\times n}$, and for all i and k $(i, k = 1, \cdots, n)$,
$$\sum_{j=1}^{n} \alpha_j \beta_i^{jk} \left(s_{ik}^* - s_{ik}\right)^T A_i \left(E_j^i s_j^*\right) \geq 0. \quad (12)$$
2. Stability condition: For all $s \neq s^*$ such that the equality holds in the equilibrium condition, and for all i $(i = 1, \cdots, n)$,
$$\sum_{j=1}^{n} \alpha_j \left\{E_i^j \left(s_i^* - s_i\right)\right\}^T A_i \left(E_j^i s_j\right) > 0. \quad (13)$$

Proof Since Eq. (10) holds for all sufficiently small ϵ_j, there exist ϵ and ϵ_j such that $\epsilon = \epsilon_1 = \cdots = \epsilon_n$ and Eq. (11) holds. Thus, an extended strategy combination $s^* \in S^{n\times n}$ is an ESS if and only if the following two conditions hold:

1'. For all $s \in S^{n\times n}$ and all i $(i = 1, \cdots, n)$,
$$\sum_{j=1}^{n} \alpha_j \left(E_i^j s_i^*\right)^T A_i \left(E_j^i s_j^*\right) \geq \sum_{j=1}^{n} \alpha_j \left(E_i^j s_i\right)^T A_i \left(E_j^i s_j^*\right). \quad (14)$$

2. The stability condition of Theorem 2.

Next, we prove that the condition 1' holds if and only if the condition 1 holds. By Eq. (14), we have

$$\sum_{j=1}^{n} \alpha_j \left\{E_i^j \left(s_i^* - s_i\right)\right\}^T A_i \left(E_j^i s_j^*\right)$$
$$= \sum_{k=1}^{n} \left\{\sum_{j=1}^{n} \alpha_j \beta_i^{jk} \left(s_{ik}^* - s_{ik}\right)^T A_i \left(E_j^i s_j^*\right)\right\} \geq 0. \quad (15)$$

Obviously, if the condition 1 holds, then the condition 1' holds by the Eq. (15).

Let $s^* \in S^{n\times n}$ be an extended strategy combination satisfying the condition 1'. Suppose that there exists $s_{i\hat{k}} \in S$ which does not satisfy Eq. (12), that is, there exists $\hat{k}$ which satisfies the following condition:

$$\sum_{j=1}^{n} \alpha_j \beta_i^{j\hat{k}} \left(s_{i\hat{k}}^* - s_{i\hat{k}}\right)^T A_i \left(E_j^i s_j^*\right) < 0. \quad (16)$$

We consider $s_i' = (s_{i1}^{*T}, \cdots, s_{i\hat{k}-1}^{*T}, s_{i\hat{k}}^T, s_{i\hat{k}+1}^{*T}, \cdots, s_{in}^{*T})^T$, where $s_{i\hat{k}}^{*T}$ is replaced by $s_{i\hat{k}}^T$. Substituting s_i' for s_i of left-hand side of Eq. (15), we have

$$\sum_{k=1}^{n} \left\{\sum_{j=1}^{n} \alpha_j \beta_i^{jk} \left(s_{ik}^* - s_{ik}'\right)^T A_i \left(E_j^i s_j^*\right)\right\}$$
$$= \sum_{j=1}^{n} \alpha_j \beta_i^{j\hat{k}} \left(s_{i\hat{k}}^* - s_{i\hat{k}}\right)^T A_i \left(E_j^i s_j^*\right) < 0. \quad (17)$$

However, Eq. (17) contradicts the assumption that $s^* \in S^{n\times n}$ satisfies the condition 1'. Thus, Eq. (12) holds for all k, that is, the condition 1' holds if the condition 1 holds. □

4.2 Replicator Dynamics

Suppose that individuals adopt pure strategies. Let s_{ij}^k be the proportion of individuals of P_i that adopt a pure strategy $k \in \Phi$ to the individuals whose sub-population is identified as P_j. We assume that an individual's strategy to each sub-population is selected independent of each other. Suppose that the rate of increase of individuals of P_i with a strategy k to the sub-population identified as P_j is expressed as the difference between the payoffs which an individual of P_i with a strategy k earns and the average payoff of P_i to the sub-population identified as P_j. Hence, replicator dynamics of this model is formulated as follows:

$$\dot{s}_{ij}^k = s_{ij}^k \sum_{l=1}^{n} \alpha_l \beta_i^{lj} (e^k - s_{ij})^T A_i \left(E_l^i s_l\right) = f_{ij}^k(s) \quad (18)$$

for $i, j = 1, \cdots, n$ and $k \in \Phi$, where e^k is the m-dimensional unit vector such that its kth element equals 1.

Since the concept of an ESS represents a stability of a strategy against invasions of mutant strategies, we expect that a population state corresponding to an ESS is robust to some perturbation. In conventional evolutionary games, the fact that all orbits of replicator dynamics converge to a population state corresponding to an ESS is well known and considered as an important property [7]. The following theorem corresponds to the property.

Theorem 3 If an extended strategy combination $s^* \in S^{n\times n}$ is an ESS, its population state combination s^* is a locally asymptotically stable equilibrium point of replicator dynamics (18). Moreover, if s^* is an interior ESS, then the equilibrium point is globally asymptotically stable.

Proof Let $s^* \in S$ be an ESS. Consider the following function:

$$V(s) = \sum_{i=0}^{n} \sum_{j=0}^{n} \left(- \sum_{s_{ij}^{k*} > 0} s_{ij}^{k*} \log \frac{s_{ij}^k}{s_{ij}^{k*}}\right).$$

It is obvious that $V(s^*) = 0$. Using Jensen's inequality, $V(s) \geq 0$ with equality if and only if $s = s^*$. The time derivative of $V(s)$ along orbits of Eq. (2) is as follows:

$$\dot{V}(s) = -\sum_{i=1}^{n} \left[\sum_{l=1}^{n} \alpha_l \left\{E_i^l (s_{ij}^* - s_{ij})\right\}^T A_i \left(E_l^i s_l\right)\right]. \quad (19)$$

By Definition 2, for all sufficiently small $\epsilon_j > 0$ $(j = 1, \cdots, n)$, Eq. (10) holds. Then, there exists a neighborhood U of s^* such that the following condition holds for all $\bar{s} \in U$:

$$\sum_{j=1}^{n} \alpha_j \left\{E_i^j \left(s_i^* - s_i\right)\right\}^T A_i \left(E_j^i \bar{s}_j\right) > 0. \quad (20)$$

Since $\dot{V}(\bar{s}) < 0$ holds for $\bar{s} \in U$, s^* is a locally asymptotically stable equilibrium point.

Let s^* is an interior ESS. In this case, the equality holds in the condition 1 of Theorem 2 for all $s \neq s^*$, and the condition 2 of Theorem 2 holds for all $s \neq s^*$. Then $\dot{V}(s) < 0$ holds. Thus, all orbits starting from $\mathrm{int}(S^{n\times n})$ converge to s^*, that is an equilibrium point s^* is globally asymptotically stable. □

4.3 Special Cases of Perceptions

In our model, suppose that, when the individuals play the game, they judge which sub-population the other individual belongs to, and select their strategies depending on their own subjective judgments. When individuals perceive the other individuals subjectively, their behaviors are affected by properties of their perceptions and a probability of misperceptions. In this paper, we introduced both β_i^{jk} and a matrix E_i^j to describe individuals' perceptions. In this subsection, we prove several properties of an ESS and replicator dynamics depending on individuals' perceptions.

The probability that individuals of P_i perceive P_k as P_h is assumed to equals each probability that individuals of P_j perceive P_k as P_h for each h $(h = 1, 2, \cdots, n)$. In this case, the perceptions of P_i to P_k equals those of P_j. From this concept, we define the equivalence of two sub-populations' perceptions as follows:

Definition 3 Perceptions of P_i and P_j to P_k are *same* if $E_i^k = E_j^k$ holds.

As a dual concept, suppose that the probability that P_i perceive P_p as P_h equals each probability that P_i perceive P_q as P_h for all h $(h = 1, 2, \cdots, n)$. In this case, the perceptions of P_i to P_p equals those to P_q in a way. Then, we introduce the following concept as follows for P_i's perceptions:

Definition 4 P_p and P_q are *indistinguishable* for P_i if $E_i^p = E_i^q$ holds.

If perceptions of P_p and P_q to P_i are same, then both P_p and P_q have the same rule of selecting their strategies to P_i. This property affects the evolution of distributions of strategies in P_i. For dynamics of strategies in P_i, we show the following theorem:

Theorem 4 We assume that perceptions of P_p and P_q to P_i are same. Let $s^*, s' \in S^{n\times n}$ be population states such that $s^* = (s_1^*, \cdots, s_n^*)$ and $s' = (s_1^*, \cdots, s_{p-1}^*, s_p', s_{p+1}^*, \cdots, s_{q-1}^*, s_q', s_{q+1}^*, \cdots, s_n^*)$. For each j $(j = 1, \cdots, n)$ and all $k \in \Phi$, $f_{ij}^k(s^*) = f_{ij}^k(s')$ holds if s_p' and s_q' satisfy the following equation:

$$\alpha_p \beta_i^{pj} s_p^* + \alpha_q \beta_i^{qj} s_q^* = \alpha_p \beta_i^{pj} s_p' + \alpha_q \beta_i^{qj} s_q'. \quad (21)$$

Proof We assume that perceptions of P_p and P_q to P_i are same. Then, $E_p^i = E_q^i$. By Eq. (18), we have

$$f_{ij}^k(s) = s_{ij}^k \sum_{l\neq p,q} \alpha_l \beta_i^{lj} (e^k - s_{ij})^T A_i \left(E_l^i s_l\right) + s_{ij}^k (e^k - s_{ij})^T A_i \left\{E_p^i \left(\alpha_p \beta_i^{pj} s_p + \alpha_q \beta_i^{qj} s_q\right)\right\}. \quad (22)$$

Thus, for each j $(j = 1, \cdots, n)$ and all $k \in \Phi$, $f_{ij}^k(s^*) = f_{ij}^k(s')$ holds if s_p' and s_q' satisfy Eq. (21). □

Using Theorem 3, we have the following theorem for an equilibrium point of replicator dynamics (18):

Theorem 5 We assume that perceptions of P_p and P_q to all sub-populations are same. Let $s^*, s' \in S^{n\times n}$ be population states such that $s^* = (s_1^*, \cdots, s_n^*)$ and $s' = (s_1^*, \cdots, s_{p-1}^*, s_p', s_{p+1}^*, \cdots, s_{q-1}^*, s_q', s_{q+1}^*, \cdots, s_n^*)$. If s_p' and s_q' satisfy Eq. (21) for all i and j $(i, j = 1, \cdots, n)$, and s^* is an equilibrium point of replicator dynamics (18), then s' is also an equilibrium point of Eq. (18).

Proof From the proof of Theorem 5, Eq. (22) holds. Since $s^* \in S^{n\times n}$ is an equilibrium point of replicator dynamics (18), when Eq. (21) holds for all i and j, we have $f_{ij}^k(s^*) = f_{ij}^k(s') = 0$ for all i, j, and k. Thus, s' is also an equilibrium point of Eq. (18). □

Definition 4 is a dual concept of Definition 3. If P_p and P_q are indistinguishable for P_i, then P_i use the same strategy to both of P_p and P_q. This property affects the evolution of distributions of strategies in P_i. For dynamics of strategies in P_i, we show the following theorem:

Theorem 6 We assume that P_p and P_q are indistinguishable for P_i. Let $s^*, s' \in S^{n\times n}$ be a population states such that $s^* = (s_1^*, \cdots, s_n^*)$ and $s' = (s_1^*, \cdots, s_{p-1}^*, s_p', s_{p+1}^*, \cdots, s_{q-1}^*, s_q', s_{q+1}^*, \cdots, s_n^*)$. For all j $(j = 1, \cdots, n)$ and $k \in \Phi$, $f_{ij}^k(s^*) = f_{ij}^k(s')$ holds if s_p' and s_q' satisfy the following equation:

$$\alpha_p E_p^i s_p^* + \alpha_q E_q^i s_q^* = \alpha_p E_p^i s_p' + \alpha_q E_q^i s_q'. \quad (23)$$

Proof We assume that P_p and P_q are indistinguishable for P_i. Then, $\beta_i^{pj} = \beta_i^{qj}$ for all j $(j = 1, \cdots, n)$. By Eq. (18), we have

$$f_{ij}^k(s) = s_{ij}^k \sum_{l\neq p,q} \alpha_l \beta_i^{lj} (e^k - s_{ij})^T A_i \left(E_l^i s_l\right) + s_{ij}^k (e^k - s_{ij})^T A_i \left\{\beta_i^{pj} \left(\alpha_p E_p^i s_p + \alpha_q E_q^i s_q\right)\right\}. \quad (24)$$

Thus, for all j $(j = 1, \cdots, n)$ and $k \in \Phi$, $f_{ij}^k(s^*) = f_{ij}^k(s')$ holds if s_p' and s_q' satisfy Eq. (23). □

Using Theorem 3, we have the following theorem for an equilibrium point of replicator dynamics (18):

Theorem 7 We assume that P_p and P_q are indistinguishable for all sub-populations. Let $s^*, s' \in S^{n\times n}$ be population states such that $s^* = (s_1^*, \cdots, s_n^*)$ and $s' = (s_1^*, \cdots, s_{p-1}^*, s_p', s_{p+1}^*, \cdots, s_{q-1}^*, s_q', s_{q+1}^*, \cdots, s_n^*)$. If s_p' and s_q' satisfy Eq. (23) for all i $(i = 1, \cdots, n)$, and s^* is an equilibrium point of replicator dynamics (18), then s' is also an equilibrium point of (18).

Proof From the proof of Theorem 6, Eq. (24) holds. Since $s^* \in S^{n \times n}$ is an equilibrium point of replicator dynamics (18), when Eq. (23) holds for all i, we have $f_{ij}^k(s^*) = f_{ij}^k(s') = 0$ for all i, j, and k. Thus, s' is also an equilibrium point of (18). □

Theorems 5 and 7 show that there is no isolated equilibrium point when individuals' perception have the special properties. There is no advantage of selecting a strategy depending on a sub-population the other individual belongs to.

Moreover, when every pair of sub-populations is indistinguishable for all sub-populations, we have the following theorem for an ESS of the model of Section 3 and replicator dynamics (18):

Theorem 8 Let a mixed strategy combination $\hat{s}^* \in S^n$ be an ESS defined by Definition 1. If every pair of sub-populations is indistinguishable for all sub-populations, then an extended strategy combination $s^* \in S^{n \times n}$ which consists of $s_i^* \in S_i$ satisfying $E_i s_i^* = \hat{s}_i^*$ $(i = 1, \cdots, n)$ is an equilibrium point of replicator dynamics (18).

Proof By Definition 4, we have $E_i^1 = E_i^2 = \cdots = E_i^n$, for all i $(i = 1, \cdots, n)$. Suppose that $E_i = E_i^j$, $\beta_i^{\cdot k} = \beta_i^{jk}$ $(i = 1, \cdots, n, k \in \Phi)$. Since $\hat{s}^* \in S^n$ is an ESS defined by Definition 1, the condition 1 of Theorem 1 holds. Then, for all $E_i s_i \in S$, that is, for all $s_i \in S_i$ $(i = 1, 2, \cdots, n)$,

$$\begin{aligned} & \left(\hat{s}_i^* - E_i s_i\right)^T A_i \left(\sum_{j=1}^{n} \alpha_j \hat{s}_j^*\right) \\ = & \sum_{j=1}^{n} \alpha_j E_i \left(s_i^* - s_i\right)^T A_i \left(E_j s_j^*\right) \geq 0. \end{aligned} \quad (25)$$

Thus $s^* \in S^{n \times n}$ satisfies the condition 1' of the proof of Theorem 2. Since $s^* \in S^{n \times n}$ satisfies the condition 1 of Theorem 2, we have

$$\begin{aligned} & s_{ik}^* = \arg\max_{s_{ik} \in S} \sum_{j=1}^{n} \alpha_j \beta_i^{\cdot k} s_{ik}^T A_i \left(E_j s_j^*\right) \\ & \Longleftrightarrow s_{ik}^l > 0 \rightarrow \sum_{j=1}^{n} \alpha_j \beta_i^{\cdot k} (e^l - s_{ik}^*)^T A_i \left(E_j s_j^*\right) = 0. \end{aligned} \quad (26)$$

Thus, $s^* \in S^{n \times n}$ which consists of $s_i^* \in S_i$ satisfying $E_i s_i^* = \hat{s}_i^*$ $(i = 1, \cdots, n)$ is an equilibrium point of replicator dynamics (18). □

Theorem 8 shows that we do not need selecting a strategy depending on a sub-population the other individual belongs to when every pair of sub-populations is indistinguishable for all sub-populations.

5 Conclusions

In this paper, we have considered the model of multipopulation asymmetric games where individuals do not know that a population is partitioned into multiple sub-populations, and we have shown that this model is a special case of a model proposed by Taylor [5]. By introducing individuals' perceptions into this model, we have proposed a model of situations that each individual selects a strategy depending on their own erroneous perceptions. We have defined an evolutionarily stable strategy (ESS) and formulated replicator dynamics in this model, and proved several properties of the proposed model. Moreover, we have proposed two concepts describe the special case of individual's perceptions, and proved several properties relating to them.

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