DESIGN OF BURSTING IN TWO-DIMENSIONAL DISCRETE-TIME NEURON MODELS

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ABSTRACT
We will propose a design method for bursting neuron models with a specified average angular frequency and duty ratio. The proposed design method derives parameters and initial conditions in the models. We also show effectiveness of the proposed design method by simulation.

1. INTRODUCTION
Bursting in a biological neuron plays crucial roles, for example, advancing secretion of a hormone[1], and contribution to constriction of the muscles effectively[2].

Mathematical neuron models are classified into continuous-time models and discrete-time ones. In continuous-time neuron models, Izhikevich clarified mechanism of generation of the bursting from bifurcation theoretical point of view[3]. On the other hand, Kitajima et al. have observed and analyzed the bursting in a discrete-time neural network[4]. The considered neural network is constructed by a chaotic neuron model proposed by Aihara et al.[5]. However, it is difficult to observe the bursting in Aihara’s single neuron model because it has one-dimensional dynamics.

Recently, we have proposed a high-dimensional discrete-time neuron model by extending Aihara’s one[6]. The model exhibits the bursting when a Hopf bifurcation for a pair of two-periodic points occurs. We have also proposed a piecewise linear discrete-time neuron model, where a piecewise linear output function is used instead of the logistic function[7]. We have shown that the piecewise linear model also exhibits the bursting when the Hopf bifurcation for the pair of two-periodic points occurs.

Moreover, it has been pointed out that a period and duty ratio of the bursting are also important[8]. Therefore, in this paper, we will propose a design method for the proposed bursting neuron models with a specified period and duty ratio by using bifurcation theory.

This paper is organized as follows: Section 2 will introduce the considered discrete-time neuron models in this paper, Sec. 3 will propose a design method of the bursting, and Sec. 4 will show its effectiveness by simulation. Finally, we will conclude this paper in Sec. 5.

2. TWO-DIMENSIONAL DISCRETE-TIME NEURON MODELS
We consider a two-dimensional discrete-time neuron model given by
\[
\begin{align*}
    y_1(t+1) &= k_1y_1(t) + k_2y_2(t) - \alpha f(y_1(t)) + c, \\
    y_2(t+1) &= y_1(t),
\end{align*}
\]
where \( t \) is a discrete time, \((y_1(t), y_2(t))\) is an internal state, and \( k_1, k_2, \alpha, \) and \( c \) are parameters. The output from the neuron \( x(t) \) is determined by \( x(t) = f(y_1(t)) \).

In [6], the output function \( f \) is given by the following logistic function:
\[
f_1(u) = \frac{1}{1+\exp(-u/\varepsilon_1)},
\]
where \( \varepsilon_1 \) is a steepness parameter.

On the other hand, in [7], the following piecewise linear output function shown in Fig. 1 is used instead of the logistic function:
\[
f_2(u) = \begin{cases} 
0 & \text{if } u \leq -\varepsilon_2/2, \\
\frac{u}{\varepsilon_2} + \frac{1}{2} & \text{if } -\varepsilon_2/2 < u < \varepsilon_2/2, \\
1 & \text{if } \varepsilon_2/2 \leq u,
\end{cases}
\]
where \( \varepsilon_2 \) is another steepness parameter.

In this paper, the discrete-time neuron model with the logistic function \( f_1 \) is called “Model I”, and the model with the piecewise-linear function \( f_2 \) is called “Model II”.

Corresponding to Eq. (3), we partition the \( y_1-y_2 \) phase plane into the following three regions \( R_i \) (\( i = I, II, III \)):
\[
\begin{align*}
    R_I &= \{(y_1, y_2) \mid y_1 \leq -\varepsilon_2/2\}, \\
    R_{II} &= \{(y_1, y_2) \mid -\varepsilon_2/2 < y_1 < \varepsilon_2/2\}, \\
    R_{III} &= \{(y_1, y_2) \mid \varepsilon_2/2 \leq y_1\}.
\end{align*}
\]

3. DESIGN OF BURSTING
3.1. Bursting Oscillations, Duty Ratio and Period
When neuron activity alternates between a quiescent state and repetitive spiking(spiking state), the neuron activity is...
3.2. Mechanism of Bursting

There are some pairs of two-periodic points in the models. When a Hopf bifurcation for the two-periodic points occurs, there exists two pairs of non-isolated invariant closed curves surrounding each two-periodic point shown in Fig. 3. The coordinates of both two-periodic points are denoted by \((\bar{y}_1, \bar{y}_2)\) and \((\bar{y}_2, \bar{y}_1)\), where \(\bar{y}_1 > \bar{y}_2\) without loss of generality. Figure 4 is enlargement of Fig. 3 around the two-periodic point \((\bar{y}_1, \bar{y}_2)\). \(y_{th}\) in Fig. 4 is threshold for firing or non-firing, i.e., if \(y_1(t) > y_{th}\) (resp. \(y_1(t) < y_{th}\)), the neuron is firing (resp. non-firing) at the time \(t\). \(x_{th} = f(y_{th})\). On the other hand, the neuron is always non-firing on the other non-isolated closed invariant curve in Fig. 3.

3.3. Occurrence Conditions for Bursting

If there exists a pair of two-periodic points \((\bar{y}_1, \bar{y}_2)\) and \((\bar{y}_2, \bar{y}_1)\), the characteristic polynomial for the two-periodic point is given by

\[
\phi(\lambda) = \lambda^2 - (u_1 u_2 + 2k_2)\lambda + k_2^2, \tag{6}
\]

where \(u_i = k_1 - \alpha f'(\bar{y}_i)\) for \(i = 1, 2\). From Eq. (6), a Hopf bifurcation for the two-periodic points occurs if the following conditions hold:

\[
k_2 = \pm 1, \ u_1 u_2 (u_1 u_2 + 4k_2) < 0. \tag{7}
\]

In this paper, we consider the case where \(k_2 = 1\). Consequently we can rewrite Eq. (7) as follows:

\[
k_2 = 1, \ -4 < u_1 u_2 < 0. \tag{8}
\]

3.4. Average Angular Frequency

We have defined the period of the bursting in Sec. 3.1. However, strictly speaking, the period cannot be defined in both models since the internal state is quasi-periodic when the bursting occurs. Therefore, we define an “average angular frequency” instead of the period. The average angular frequency \(\omega\) is defined by a central angle shown in Fig. 5. Note that \(\omega\) is defined by the central angle between the time \(t\) and \(t + 2\) since the invariant closed curves are two-periodic.

3.5. Design

In this subsection, we propose a design method for a bursting neuron with specified average angular frequency \(\omega\) and duty ratio \(r\). Note that eigenvalues of the pair of the two-periodic points are \(e^{\pm j \omega}\), that is, \(e^{\pm j \omega}\) are roots of \(\phi(\lambda) = 0\). Therefore, by using Eq. (8), we obtain the following result:

\[
2 \cos \omega = u_1 u_2 + 2 \iff u_1 u_2 + 4 \sin^2 \frac{\omega}{2} = 0. \tag{9}
\]

If an initial condition is set to be around the two-periodic point \((\bar{y}_1, \bar{y}_2)\), then the bursting occurs and \((y_1(t), y_2(t))\) is quasi-periodic. Suppose that the non-isolated invariant closed curve is a circle with the center \((\bar{y}_1, \bar{y}_2)\) and radius \(d\) when the initial condition \((y_1(0), y_2(0))\) is set to be \((\bar{y}_1 + d, \bar{y}_2)\). Figure 4 clearly shows that the duty ratio \(r\) is given by

\[
r = \frac{\beta}{2\pi}, \tag{10}
\]

where \(\beta\) is a central angle of the dotted arc, which corresponds to the spiking state. From Eq. (10) and Fig. 4, the following relation is derived:

\[
\bar{y}_1 = y_{th} - d \cos \beta \pi. \tag{11}
\]

On the other hand, \(\bar{y}_1\) and \(\bar{y}_2\) are roots of the following equation:

\[
k_1 y - \alpha f(y) + c = 0. \tag{12}
\]

Note that the above equation holds only if \(k_2 = 1\). From Eqs. (11) and (12), the parameter \(c\) is determined by

\[
c = -k_1 (y_{th} - d \cos \beta \pi) + \alpha f(y_{th} - d \cos \beta \pi). \tag{13}
\]
Finally, we discuss how the parameters and the initial condition are determined. The parameter $k_1$ is set to be a root of Eq. (9), $k_2$ is set to be 1, $\tilde{y}_1$ is set by (11), $c$ is determined by Eq. (13), and $\tilde{y}_2$ is set to be a root of Eq. (12), which is different from $\tilde{y}_1$.

The radius $d$ can be set freely in Model I. However, in Model II, there exists a pair of non-isolated invariant closed curves shown in Fig. 3, and the left and right invariant closed curve in Fig. 3 must lie in $R_l$ and $R_r$, respectively. Therefore, $d$ is restricted. By assuming that the invariant closed curves are circles with the radius $d$, they lie in each region if the following conditions hold:

$$\tilde{y}_1 - d \geq -\frac{\varepsilon}{2}, \tilde{y}_1 + d \leq \frac{\varepsilon}{2}. \tag{14}$$

The above inequalities can be rewritten as follows:

$$0 < d \leq \min\left\{\frac{0.5\varepsilon_2 + y_{th}}{1 + \cos r\pi}, \frac{0.5\varepsilon_2 - y_{th}}{1 - \cos r\pi}\right\}. \tag{15}$$

In Model II, we set the parameter $d$ satisfying Eq. (15).

### 4. EXAMPLES

#### 4.1. Model I

We set the parameter $\alpha$, $\varepsilon_1$, $d$, and $x_{th}$ to be 1.0, 0.02, 0.04, and 0.4, respectively. Suppose that the specifications of bursting are given by

$$\omega = 0.3, r = 0.4, \tag{16}$$

then there exist the parameters $k_1$ and $c$ for the specifications. The proposed design method gives two sets of the parameter and the initial conditions since $k_1$ is determined by the quadratic equation Eq. (9). The one set is not adopted since $k_1 > 1$, and the other set is given by

$$k_1 = 0.0092, c = 0.2645, y_1(0) = 0.01953, y_2(0) = -28.7677. \tag{17}$$

Figure 6 shows numerical simulation with Eq. (17), and we get the following result:

$$\omega' = 0.2567, r' = 0.3141. \tag{18}$$

#### 4.2. Model II

We set the parameters $\alpha$, $\varepsilon_2$, and $x_{th}$ to be 1.0, 0.25, and 0.5, respectively. Suppose that the specifications of bursting are given by

$$\omega = 1.0, r = 0.3. \tag{19}$$

Then, there exist the parameters $k_1$ and $c$ for the specifications. The proposed design method gives the following result:

$$k_1 = 0.2448, c = 0.3436, 0 < d \leq d_{\text{max}} = 0.0787. \tag{20}$$

We set the radius $d$ to be $0.9d_{\text{max}}$, and the following initial condition is derived:

$$y_1(0) = 0.0292, y_2(0) = -1.4034. \tag{21}$$

Note that we have another set of the parameters and the initial conditions. However, we adopt Eqs. (20) and (21) because of the same reason described in Sec. 4.1.

Numerical simulation shown in Fig. 7 gives the following result:

$$\omega' = 0.9983, r' = 0.3259. \tag{22}$$

### 5. CONCLUDING REMARKS

In this paper, we have proposed a design method for bursting neuron models with a specified average angular frequency and duty ratio. The proposed method determines both parameters and initial conditions for the models.

However, the proposed method is not sufficient for Model I since it is based on linearization for the neighborhood of
the two-periodic points. One of our future studies is to extend the proposed method taking the nonlinearity into account.

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7. REFERENCES


